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High-order compact exponential finite difference methods for convection–diffusion type problems

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Abstract

A class of high-order compact (HOC) exponential finite difference (FD) methods is proposed for solving one- and two-dimensional steady-state convection–diffusion problems. The newly proposed HOC exponential FD schemes have nonoscillation property and yield high accuracy approximation solution as well as are suitable for convection-dominated problems. The $O(h^4)$ compact exponential FD schemes developed for the one-dimensional (1D) problems produce diagonally dominant tri-diagonal system of equations which can be solved by applying the tridiagonal Thomas algorithm. For the two-dimensional (2D) problems, $O(h^4 + k^4)$ compact exponential FD schemes are formulated on the nine-point 2D stencil and the line iterative approach with alternating direction implicit (ADI) procedure enables us to deal with diagonally dominant tridiagonal matrix equations which can be solved by application of the one-dimensional tridiagonal Thomas algorithm with a considerable saving in computing time. To validate the present HOC exponential FD methods, three linear and nonlinear problems, mostly with boundary or internal layers where sharp gradients may appear due to high Peclet or Reynolds numbers, are numerically solved. Comparisons are made between analytical solutions and numerical results for the currently proposed HOC exponential FD methods and some previously published HOC methods. The present HOC exponential FD methods produce excellent results for all test problems. It is shown that, besides including the excellent performances in computational accuracy, efficiency and stability, the present method has the advantage of better scale resolution. The method developed in this article is easy to implement and has been applied to obtain the numerical solutions of the lid driven cavity flow problem governed by the 2D incompressible Navier–Stokes equations using the stream function-vorticity formulation.

Keywords: Convection–diffusion equation; High-order compact scheme; Exponential finite difference method; Boundary layer; Stream function-vorticity formulation; Navier–Stokes equations

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1. Introduction

Numerical prediction of the convection–diffusion equation plays a very important role in computational fluid dynamics (CFD) to simulate flow problems. Therefore, accurate, stable and efficient difference representations of the convection–diffusion equations are of vital importance. It has been discovered that although the CD approximation is second-order accurate, classical iterative methods for solving the resulting linear system do not converge when the cell Reynolds number is greater than a certain value. The UD scheme is unconditional stable, but is only first-order accurate, and the resulting solution exhibits the effect of artificial viscosity [28,30]. In recent years, high order compact (HOC) FD methods have generated renewed interest and a variety of specialized techniques have been developed [3–6,9,15,17,19,20,22–24,27,32–34,37]. For the 1D convection–diffusion equations, Dekema and Schultz [6] developed higher-order methods using the Taylor series expansion. For 2D convection–diffusion equations, Gupta et al. [17] employed series expansions to the differential equation to develop a fourth-order nine-point compact FD formula. Similar HOC FD schemes have been developed and applied to the convection–diffusion equations and the incompressible Navier–Stokes equations by several authors [4,5,9,15,22,32,33]. Dennis and Hudson [9] developed the same scheme as in Ref. [17] using another approach. Recently, Kolesnikov and Baker [20] developed and tested a new approach to designing high order, defined to exceed a third accurate method.

All of the above HOC schemes mentioned may be classified as high-order compact polynomial FD schemes; i.e., the influence coefficients of the FD formulation are connected to polynomial functions of the coefficients of the differential operator. The algebraic manipulations for deriving these HOC polynomial FD schemes are complicated, interested readers are referred to Refs. [17,22,32,33]. The 4OC polynomial FD schemes have good numerical stability and yield higher accuracy approximations [9,17]. For convection-dominated problems or large cell Reynolds number Re_c case, however, analyses and numerical tests given by some researchers [3,37,40] have showed that HOC polynomial FD schemes may yield nonphysical spurious behavior because no maximum principle is guaranteed and/or no upwind effect is preserved. Thus, the existing HOC polynomial FD schemes are not suitable for particular physical problems, such as abrupt boundary layer in convection-dominated problems and shock-like discontinuities caused by local nonlinearities, unless a very fine mesh is used [19,20]. This dilemma can be resolved by utilizing nonuniform mesh and local mesh refinement strategies [4,5,12,13,27,30]. Unfortunately, the boundary layer location or the singularity region must be known in this case.

An alternative approach to HOC polynomial FD methods is the class of HOC exponential FD methods, i.e., the coefficients of the FD formulation are connected to exponential functions of the coefficients of the differential operator. The exponential FD scheme has the noteworthy features that upwind effect is inherently considered in the exponential functions and coefficient matrix is diagonally dominant unconditionally, which makes it very suitable for singular perturbation problems characterized by boundary and/or transition layers where the gradients of the solution are large. The exponential FD scheme was first introduced by Allen and Southwell [1] to solve the second-order partial differential equation governing the transport of vorticity. The methods for the one-dimensional steady, linear convection–diffusion equation have been studied in [2,10,21]. MacKinnon and Johnson [24] derived an 4OC exponential FD scheme for the 2D convection–diffusion equation with constant convection coefficients. Recently, Radhakrishna Pillai [27] developed 4OC exponential FD methods for solving 1D and 2D convection–diffusion equations with constant and variable convection coefficients on compact stencils. However, the coefficient matrix of schemes may fail to be diagonally dominant for the convection-dominated problems with variable convection coefficients, since the influence coefficients involve both exponential and polynomial functions in the FD approximations. In [3], a perturbational 4OC exponential FD scheme with diagonally dominant coefficient matrix has been developed for the convection–diffusion equations based on a second-order exponential FD scheme proposed by Chen et al. [3]. However, it is easily found that the influence of source term in the FD equation established in [3] becomes less and less as the cell Reynolds number Re_c increased, and eventually for very large Re_c value, this influence becomes negligible. The drawbacks of the exponential FD methods proposed in [3,27] will be illustrated in detail in the numerical examples.

This paper is primarily aimed at developing 4OC exponential FD schemes with diagonally dominant coefficient matrix for solving the steady convection–diffusion equations. The newly proposed HOC exponential

FD schemes have nonoscillation property and yield high accuracy approximation solution as well as are suitable for convection-dominated problems in presence or absence of source term. To obtain the HOC exponential FD scheme for the 1D convection–diffusion equation with constant convection coefficient, the idea of modified differential equation analysis of Warming and Hyett [36] is employed to determine the computational stencil coefficients appropriate for the desired order of accuracy in our study. The construction of the 4OC exponential FD scheme for the 2D problem with constant convection coefficients are based on the 4OC exponential FD scheme proposed for the 1D one in this paper. For the variable convection coefficient problems, 4OC compact exponential FD schemes are achieved based on the 4OC exponential FD schemes proposed for the constant convection coefficient problems and a practical technique, named reminder term modification approach [23]. The currently developed HOC exponential FD schemes are applied to three linear and nonlinear problems for which numerical and analytical results are available, mostly with boundary or internal layers where sharp gradients may appear due to high Peclet or Reynolds numbers. As the basis of a discretization method for the incompressible, 2D, steady-state flow problems, the 4OC exponential FD formulation proposed for the 2D convection–diffusion equation is applied to the stream function-vorticity formulation of the Navier–Stokes equations and the lid driven cavity flow problem is solved.

In the next section, we first present a three-point $O(h^4)$ compact exponential FD scheme for the 1D convection–diffusion equation with constant convection coefficient; then we develop a three-point 4OC exponential FD scheme of the 1D problem with variable convection coefficient. In Section 3, $O(h^4 + k^4)$ compact exponential FD schemes on a 3×3 stencil are proposed for the 2D convection–diffusion equations with constant and variable convection coefficients. Applications of the newly proposed 4OC formulation to the Navier–Stokes equations in the stream function-vorticity formulation follow in Section 4. To validate the feasibility of the proposed HOC exponential FD methods, numerical experiments for one- and two-dimensional convection–diffusion equations and the numerical solutions of the lid driven cavity flow problem are performed in Section 5. Finally, Section 6 is devoted to some concluding remarks.

2. High-order compact exponential FD methods: 1D case

Consider the steady one-dimensional nonhomogeneous convection–diffusion model problem

$$-au_{xx} + c(x)u_x = f(x) \quad x \in [\mathbf{0}, \mathbf{1}] \tag{1}$$

where a is the constant conductivity, c is the convective velocity which might be a constant or vary spatially, f is a sufficiently smooth function of x, and u may represent heat, vorticity, etc. This equation is consistent with singular-perturbation problem as a is a small parameter.

To set up the difference equation of (1) divide $[\mathbf{0}, \mathbf{1}]$ into N equal parts with $x_i = ih$, $h = x_{i+1} - x_i$, $u_i = u(x_i)$, $c_i = c(x_i)$, $f_i = f(x_i)$ and $i \in \{0, 1, 2, \ldots, N\}$. For a sufficiently smooth solution u, derivatives in (1) at interior grid points x_i, can be defined using Taylor's theorem as

$$u_{x_i} = D_h u_i - \sum_{n=1}^{\infty} \frac{h^{2n}}{(2n+1)!} D_x^{2n+1} u_i \tag{2}$$

$$u_{xx_i} = D_h^2 u_i - \sum_{n=1}^{\infty} \frac{2h^{2n}}{(2n+2)!} D_x^{2n+2} u_i \tag{3}$$

where $D_h u_i = (u_{i+1} - u_{i-1})/(2h)$ and $D_h^2 u_i = (u_{i+1} - 2u_i + u_{i-1})/h^2$ are the central difference approximations for the first and second derivatives and D_x^n is the nth-order exact derivative operator at any interior x.

In this section, we will develop several HOC exponential FD schemes for solving the convection–diffusion equation (1). To formulate these schemes, we first introduce an $O(h^2)$ compact exponential FD method for Eq. (1). In the subdomain $[x_{i-1}, x_{i+1}]$, let us rewrite Eq. (1) as

$$-a\mathrm{e}^{\frac{c_i x}{a}} \left(\mathrm{e}^{-\frac{c_i x}{a}} u_x \right)_x = f_i$$

namely,

$$-a\left(\mathrm{e}^{-\frac{c_i x}{a}}u_x\right)_x = f_i \mathrm{e}^{-\frac{c_i x}{a}} \tag{4}$$

in which, we have assumed that $c = c_i$ and $f = f_i$ in the subdomain $[x_{i-1}, x_{i+1}]$. Integration of Eq. (4) over a spaced interval from $x_{i-\frac{1}{2}}$ to $x_{i+\frac{1}{2}}$ gives

$$c_i\left[\mathrm{e}^{-\frac{c_i h}{2a}}(u_x)_{i+\frac{1}{2}} - \mathrm{e}^{\frac{c_i h}{2a}}(u_x)_{i-\frac{1}{2}}\right] = \left(\mathrm{e}^{-\frac{c_i h}{2a}} - \mathrm{e}^{\frac{c_i h}{2a}}\right)f_i \tag{5}$$

Using the central difference to approximate u_{xi}, we get

$$u_{xi} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h} + \mathrm{O}(h^2)$$

and thus, we have

$$-\alpha_i D_h^2 u_i + c_i D_h u_i = f_i \tag{6}$$

where

$$\alpha_i = \begin{cases} \frac{c_i h}{2}\coth\left(\frac{c_i h}{2a}\right), & c_i \neq 0 \\ a, & c_i = 0 \end{cases} \tag{7}$$

Eq. (6) is called as a second-order exponential FD scheme for the convective diffusion model problem (1), which is nodally exact and gives rise to a diagonally dominant tri-diagonal system of equations. In addition, scheme (6) provides the exact solution for the 1D convection–diffusion equation with constant convection coefficient in the absence of a source term. Scheme (6) and its some variants have been proposed via other approaches [1,27–29].

It is easily found that the second-order exponential FD scheme (6) applied to Eq. (1) is equivalent to the standard second-order central FD formula applied to the following equation:

$$-\frac{ch}{2}\coth\left(\frac{ch}{2a}\right)u_{xx} + cu_x = f \tag{8}$$

Eq. (8) also shows that, when the second-order exponential FD scheme (6) is used, an artificial diffusion coefficient $a[\frac{ch}{2a}\coth(\frac{ch}{2a}) - 1]$ is perturbed to Eq. (1).

2.1. $O(h^4)$ compact exponential FD method: constant coefficient case

Consider the FD scheme for Eq. (1) with constant convection coefficient at a grid point x_i as

$$-\alpha D_h^2 u_i + cD_h u_i = c_0 f_i + c_1 f_{xi} + c_2 f_{xxi} \tag{9}$$

where

$$\alpha = \begin{cases} \frac{ch}{2}\coth\left(\frac{ch}{2a}\right), & c \neq 0 \\ a, & c = 0 \end{cases} \tag{10}$$

D_h^2 and D_h are as defined previously. In order to determine the parameters c_0, c_1 and c_2, let us rewrite Eq. (10) as

$$-\alpha D_h^2 u_i + cD_h u_i = c_0(-au_{xx} + cu_x)_i + c_1(-au_{xx} + cu_x)_{xi} + c_2(-au_{xx} + cu_x)_{xxi} \tag{11}$$

Straightforwardly calculating the right-hand side of Eq. (11), and substituting (2) and (3) into (11) and rearranging it, we obtain the following modified differential equation corresponding to the scheme (11):

$$\begin{aligned} -au_{xx} + cu_x = {} & f_i + (c_0 - 1)cu_{xi} + (-c_0 a + c_1 c + \alpha)u_{xxi} + \left(-c_1 a + c_2 c - \frac{ch^2}{6}\right)D_x^3 u_i \\ & + \left(\frac{h^2}{12}\alpha - c_2 a\right)D_x^4 u_i + \mathrm{O}(h^4) \end{aligned} \tag{12}$$

Letting

$$(c_0 - 1)c = 0, \quad -c_0 a + c_1 c + \alpha = 0, \quad -c_1 a + c_2 c - \frac{ch^2}{6} = 0 \tag{13}$$

and solving the above resulting equations, we get the parameters

$$c_0 = 1, \quad c_1 = \begin{cases} \frac{a-\alpha}{c}, & c \neq 0, \\ 0, & c = 0, \end{cases} \quad c_2 = \begin{cases} \frac{a(a-\alpha)}{c^2} + \frac{h^2}{6}, & c \neq 0 \\ \frac{h^2}{12}, & c = 0 \end{cases} \tag{14}$$

This scheme gives rise to a diagonally dominant tri-diagonal system of equations. The Taylor-series truncation errors analysis show that Eq. (9) with (14) for solving the model problem (1) is an $O(h^4)$ compact FD scheme. Notice that second-order central differences of the derivatives of f may be used in Eq. (9) while still maintaining overall $O(h^4)$ accuracy on 3-point stencil. In Section 5, we shall present the numerical results that verify this conclusion.

It is interesting to note that the 4OC exponential FD scheme (9) for the model Eq. (1) is actually the standard second-order central FD scheme for the following ordinary differential equation

$$-\alpha u_{xx} + c u_x = f + c_1 f_x + c_2 f_{xx} \tag{15}$$

We see that Eq. (15) is a perturbation of Eq. (1) in the sense that an artificial diffusion coefficient $a[\frac{ch}{2a}\coth(\frac{ch}{2a}) - 1]$ and an artificial source term $c_1 f_x + c_2 f_{xx}$ have been added.

2.2. $O(h^4)$ compact exponential FD method: variable coefficient case

Consider the FD scheme for Eq. (1) with variable convection coefficient at a grid point x_i as

$$-\alpha_i D_h^2 u_i + c_i D_h u_i = f_i + c_1 f_{xi} + c_2 f_{xxi} \tag{16}$$

in which

$$\alpha_i = \begin{cases} \frac{c_i h}{2}\coth\left(\frac{c_i h}{2a}\right), & c_i \neq 0 \\ a, & c_i = 0 \end{cases}$$
$$c_1 = \begin{cases} \frac{a-\alpha_i}{c_i}, & c_i \neq 0, \\ 0, & c_i = 0, \end{cases} \quad c_2 = \begin{cases} \frac{a(a-\alpha_i)}{c_i^2} + \frac{h^2}{6}, & c_i \neq 0 \\ \frac{h^2}{12}, & c_i = 0 \end{cases} \tag{17}$$

Using the Taylor series expansions and the original differential equation (1), we derive the following modified differential equation corresponding to the scheme (16):

$$-a u_{xx} + c u_x - 2c_2 c_x u_{xx} - (c_1 c_x + c_2 c_{xx}) u_x + O(h^4) = f \tag{18}$$

where c_1 and c_2 are given by Eq. (17) and $c_1 = -\frac{c_i h^2}{12a} + O(h^4)$, $c_2 = \frac{h^2}{12} + O(h^4)$.

Eq. (18) shows that the local truncation error of Eq. (16) is only of $O(h^2)$. In order to obtain an $O(h^4)$ scheme, adding the term $2c_2 c_x u_{xx} + (c_1 c_x + c_2 c_{xx}) u_x$ to the left-hand side of Eq. (1) and neglecting the terms of fourth order, we get

$$-A_f u_{xx} + C_f u_x = f \tag{19}$$

where $A_f = a - 2c_2 c_x$ and $C_f = c + c_1 c_x + c_2 c_{xx}$ with c_1 and c_2 given by Eq. (17) or $A_f = a - \frac{h^2 c_x}{6} + O(h^4)$ and $C_f = c - \frac{h^2}{12}(\frac{c c_x}{a} - c_{xx}) + O(h^4)$. Actually, to derive HOC FD approximations for convection and diffusion problems, this technique have been used by several authors [19,27,23]. This technique was called as remainder term modification approach in [23].

Eqs. (19) and (1) are the same in the form. Being similar to scheme (9), an 4OC exponential FD scheme for the convective diffusion model problem (1) is given by

$$-\Lambda_i D_h^2 u_i + C_{fi} D_h u_i = f_i + C_1 f_{xi} + C_2 f_{xxi} \tag{20}$$

where

$$\Lambda_i = \begin{cases} \frac{C_{fi}h}{2}\coth\left(\frac{C_{fi}h}{2A_{fi}}\right), & C_{fi} \neq 0 \\ A_{fi}, & C_{fi} = 0 \end{cases}$$
$$C_1 = \begin{cases} \frac{A_{fi}-\Lambda_i}{C_{fi}}, & C_{fi} \neq 0, \\ 0, & C_{fi} = 0, \end{cases} \quad C_2 = \begin{cases} \frac{A_{fi}(A_{fi}-\Lambda_i)}{C_{fi}^2} + \frac{h^2}{6}, & C_{fi} \neq 0 \\ \frac{h^2}{12}, & C_{fi} = 0 \end{cases} \tag{21}$$

and A_f and C_f are as defined earlier.

This scheme produces a diagonally dominant tri-diagonal system of equations. The Taylor-series truncation errors analysis show that Eq. (20) with (21) is an 4OC exponential FD scheme for the convection–diffusion Eq. (1). Notice that central differences of second-order accuracy for the derivatives of c and f may be used in equation (20) with (21) while still maintaining overall fourth-order accuracy on 3-point stencil.

It is easily verified that the 4OC exponential FD scheme (20) applied to Eq. (1) is equivalent to the second-order central difference scheme applied to the following equation

$$-\Lambda u_{xx} + C_f u_x = f + C_1 f_x + C_2 f_{xx} \tag{22}$$

Thus when the O(h^4) FD scheme (20) is used, Eq. (1) is artificially perturbed to the above equation. An artificial diffusion coefficient $a\left[\frac{C_f h}{2a}\coth(\frac{C_f h}{2A_f}) - 1\right]$, an artificial convection coefficient $c_1 c_x + c_2 c_{xx}$ and an artificial source term $C_1 f_x + C_2 f_{xx}$ have been added.

2.3. O(h^4) compact exponential FD schemes: monotonicity and comparison

The 4OC exponential FD scheme (20) may be written as

$$\frac{C_{fi}}{h\left(e^{\frac{C_{fi}h}{2A_{fi}}} - e^{-\frac{C_{fi}h}{2A_{fi}}}\right)}\left(-e^{\frac{C_{fi}h}{2A_{fi}}}u_{i-1} + 2\cosh\left(\frac{C_{fi}h}{2A_{fi}}\right)u_i - e^{-\frac{C_{fi}h}{2A_{fi}}}u_{i+1}\right) = F_i \tag{23}$$

where $F_i = f_i + C_1 f_{xi} + C_2 f_{xxi}$.

A distinguishing feature of the 4OC exponential FD scheme (23) is its coefficient matrix $A = (a_{ij})$ is a real, irreducible diagonally dominant matrix with $a_{ij} \leqslant 0$ for $i \neq j$ and $a_{ii} > 0$. Referring to Appendix 1, we conclude that A is an M-matrix, and $A^{-1} > 0$ holds. Under this condition, the solutions computed from the M-matrix equation are unconditionally monotonic. By means of the M-matrix theory [25], there is a potential advantage in using the currently proposed scheme to resolve any possible sharp gradient in the flow. In Section 5, we shall present the numerical results that verify this inference.

For the convection diffusion model problem (1), the perturbational 4OC exponential FD scheme of Chen et al. [3] can also be written as

$$\frac{2}{h^2}\cosh\left(\frac{c_{pi}h}{2a}\right)u_i = \frac{1}{h^2}\left(e^{-\frac{c_{pi}h}{2a}}u_{i+1} + e^{\frac{c_{pi}h}{2a}}u_{i-1}\right) + f_{pi}/a \tag{24}$$

where

$$c_p = c + \frac{h^2}{12}\left(\frac{cc_x}{a} + c_{xx}\right)$$
$$f_p = f + \frac{h^2}{12}\left[2\left(\left(\frac{c}{2a}\right)^2 + 2\frac{c_x}{2a}\right)f - 2\frac{c}{2a}f_x + f_{xx}\right] \tag{25}$$

Scheme (23) may also be written as

$$-\alpha_{pi}D_h^2 u_i + c_{pi}D_h u_i = \frac{c_{pi}h}{e^{\frac{c_{pi}h}{2a}} - e^{-\frac{c_{pi}h}{2a}}}f_{pi}/a \tag{26}$$

in which

$$\alpha_{pi} = \begin{cases} \frac{c_{pi}h}{2}\coth\left(\frac{c_{pi}h}{2a}\right), & c_{pi} \neq 0 \\ a, & c_{pi} = 0 \end{cases} \tag{27}$$

Referring to Appendix 1, we conclude easily from Eq. (24) that the coefficient matrix A is an M-matrix, and $A^{-1} > 0$ holds, thus, the solutions computed from the M-matrix equation are unconditionally monotonic. Unfortunately, it is clearly seen from (26) that the influence of the source term becomes less and less as the cell Reynolds number $Re_c = c_{pi}h$ increased, and eventually for very large Re_c value, its influence becomes negligible. This indicates the compact exponential FD scheme (24) is not applicable for the convection-dominated problems with nonzero source term on coarse mesh. The drawback of the 4OC exponential FD schemes proposed in [3] will be illustrated in detail in the numerical examples.

Using the 4OC exponential FD method of Radhakrishna Pillai [27] to Eq. (1) with variable convection coefficients, yields

$$-AD_h^2u_i + CD_hu_i = f_i + c_1D_hf_i + c_2D_h^2f_i \tag{28}$$

where

$$A = \alpha_i - \frac{h^2}{6}D_hc_i, \quad C = c_i - \frac{h^2}{12}\left(\frac{c_i}{a}D_hc_i - D_h^2c_i\right) \tag{29}$$

and α_i, c_1 and c_2 are given by Eq. (17). Here, it must be pointed out that Eq. (28) is the same as the scheme proposed in [27]. Eq. (28) can also be written as

$$\frac{2A}{h^2}u_i = \left(\frac{A}{h^2} + \frac{C}{2h}\right)u_{i-1} + \left(\frac{A}{h^2} - \frac{C}{2h}\right)u_{i+1} + f_i + c_1D_hf_i + c_2D_h^2f_i \tag{30}$$

Obviously, the influence coefficients of Eq. (28) involve both exponential and polynomial functions. For the convection diffusion problems with constant convection coefficients, it is easy to find that the coefficients of matrix of Eq. (30) is M-matrix. For the convection diffusion problems with variable convection coefficients, however, the discretization equation (30) may not obey the positive coefficient rule [26]. Consequently, unstable numerical results or nonphysical spurious oscillation may occur on coarse mesh. We shall verify this issue through numerical results in Section 5.

3. Extension to two-dimensions case

In this section we will extend the above 4OC exponential FD formulation and method for the 1D convection diffusion problems to 2D ones. Consider the two-dimensional nonhomogeneous convective diffusion model problem

$$-au_{xx} - bu_{yy} + c(x,y)u_x + d(x,y)u_y = f(x,y) \tag{31}$$

where a and b are constants and c and d vary spatially, and f is a sufficiently smooth function with respect to x and y. This equation is consistent with the 2D steady-state Navier–Stokes equations for constant viscosity. Let the step-length in the x-direction be $h = 1/n$ and in the y-direction be $k = 1/m$, where n and m are the numbers of subdivisions in the x- and y-directions respectively.

Following [34], the 2D convection–diffusion equation (31) can be separated into the following two equations:

$$\begin{cases} -au_{xx} + c(x,y)u_x = f_1(x,y) \\ f_1 = f(x,y) + bu_{yy} - d(x,y)u_y \end{cases} \tag{32}$$

and

$$\begin{cases} -bu_{yy} + d(x,y)u_y = f_2(x,y) \\ f_2 = f(x,y) + au_{xx} - c(x,y)u_x \end{cases} \tag{33}$$

Applying (6) to the 1D-like Eqs. (32) and (33) respectively, we have

$$-\alpha_h D_h^2 u_{ij} + c_{ij} D_h u_{ij} = f_{1ij} \tag{34}$$

and

$$-\alpha_k D_k^2 u_{ij} + d_{ij} D_k u_{ij} = f_{2ij} \tag{35}$$

where $D_h u_{ij} = (u_{i+1,j} - u_{i-1,j})/(2h)$, $D_h^2 u_{ij} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2$, $D_k u_{ij} = (u_{i,j+1} - u_{i,j+1})/(2k)$ and $D_k^2 u_{ij} = (u_{i,j+1} - 2u_{i,j} + u_{i,j-1})/k^2$ are the central difference approximations for the first and second derivatives with respect to x and y respectively.

Adding (34) to (35) and using (31) yields

$$-\alpha_h D_h^2 u_{ij} - \alpha_k D_k^2 u_{ij} + c_{ij} D_h u_{ij} + d_{ij} D_k u_{ij} = f_{ij} \tag{36}$$

where

$$\alpha_h = \begin{cases} \frac{c_{ij}h}{2} \coth\left(\frac{c_{ij}h}{2a}\right), & c_{ij} \neq 0, \\ a, & c_{ij} = 0, \end{cases} \quad \alpha_k = \begin{cases} \frac{d_{ij}k}{2} \coth\left(\frac{d_{ij}k}{2b}\right), & d_{ij} \neq 0 \\ b, & d_{ij} = 0 \end{cases} \tag{37}$$

Eq. (36) with (37) is an $O(h^2 + k^2)$ compact exponential FD approximation on 2D five-point stencil for the 2D convection–diffusion equations at the point (x_i, y_j).

It is easily verified that the $O(h^2 + k^2)$ scheme (36) applied to Eq. (31) is equivalent to the standard second-order central FD formulae applied to the following partial differential equation at the point (x_i, y_j):

$$-\alpha_h u_{xx} - \alpha_k u_{yy} + cu_x + du_y = f \tag{38}$$

Eq. (38) also shows that, when the compact FD scheme (36) is used, artificial diffusion coefficients $a\left[\frac{ch}{2a}\coth(\frac{ch}{2a}) - 1\right]$ and $b\left[\frac{dk}{2b}\coth(\frac{dk}{2b}) - 1\right]$ are perturbed to Eq. (31).

3.1. $O(h^4 + k^4)$ compact exponential FD method: constant coefficients case

In this subsection, we use the above approach to derive an $O(h^4 + k^4)$ compact exponential FD scheme for the model problem (31) with constant coefficients. Scheme (9), established for the 1D constant coefficient case, can be extended to the 2D case.

Applying (9) to one-dimensional-like Eqs. (32) and (33) respectively, we obtain

$$-\alpha_h D_h^2 u_{ij} + cD_h u_{ij} = F_{1ij} \tag{39}$$

$$-\alpha_k D_k^2 u_{ij} + dD_k u_{ij} = F_{2ij} \tag{40}$$

where $F_{1ij} = f_{1ij} + c_1 f_{1xij} + c_2 f_{1xxij}$ and $F_{2ij} = f_{2ij} + d_1 f_{2yij} + d_2 f_{2yyij.}$

Straightforwardly calculating f_{1x}, f_{1xx}, f_{1y} and f_{2yy} and substituting into the right-hand sides of (39) and (40), and then combining (39) and (40) and rearranging, we obtain an $O(h^4 + k^4)$ compact exponential FD approximation to (31) at a mesh point (x_i, y_j) as

$$\left(-\alpha_h D_h^2 - \alpha_k D_k^2 + cD_h + dD_k + ED_h D_k + GD_k^2 D_h + HD_h^2 D_k + KD_k^2 D_h^2\right) u_{ij} = F_{ij} \tag{41}$$

where

$$\begin{aligned} &\alpha_h = \begin{cases} \frac{ch}{2} \coth\left(\frac{ch}{2a}\right), & c \neq 0, \\ a, & c = 0, \end{cases} \quad \alpha_k = \begin{cases} \frac{dk}{2} \coth\left(\frac{dk}{2b}\right), & d \neq 0 \\ b, & d = 0 \end{cases} \\ &c_1 = \begin{cases} \frac{a-\alpha_h}{c}, & c \neq 0, \\ 0, & c = 0, \end{cases} \quad c_2 = \begin{cases} \frac{a(a-\alpha_h)}{c^2} + \frac{h^2}{6}, & c \neq 0 \\ \frac{h^2}{12}, & c = 0 \end{cases} \\ &d_1 = \begin{cases} \frac{b-\alpha_k}{d}, & d \neq 0, \\ 0, & d = 0, \end{cases} \quad d_2 = \begin{cases} \frac{b(b-\alpha_k)}{d^2} + \frac{k^2}{6}, & d \neq 0 \\ \frac{k^2}{12}, & d = 0 \end{cases} \\ &E = c_1 d + d_1 c, \quad G = d_2 c - c_1 b, \quad H = c_2 d - d_1 a, \quad K = -c_2 b - d_2 a \\ &F = f + c_1 f_x + c_2 f_{xx} + d_1 f_y + d_2 f_{yy} \end{aligned} \tag{42}$$

Eq. (41) with (42) is a nine-point compact exponential FD scheme of $O(h^4 + k^4)$ for the 2D convection–diffusion model problem (31) with constant convection coefficients. The accuracy order of scheme (41) shall be shown by the numerical results in Section 5.

It is easily found that the $O(h^4 + k^4)$ compact exponential FD scheme (41) applied to Eq. (31) is equivalent to the standard second-order central FD formulae to the following partial differential equation at the point (x_i, y_j):

$$-\alpha_h u_{xx} - \alpha_k u_{yy} + c u_x + d u_y = F - E u_{xy} - G u_{xyy} - H u_{xxy} - K u_{xxyy} \tag{43}$$

We see that, when the $O(h^4 + k^4)$ compact exponential FD scheme (41) is used, Eq. (31) is artificially perturbed to the above Eq. (43). Artificial diffusion coefficients $a\left[\frac{ch}{2a}\coth(\frac{ch}{2a}) - 1\right]$, $b\left[\frac{dk}{2b}\coth(\frac{dk}{2b}) - 1\right]$ and an artificial source term $c_1 f_x + c_2 f_{xx} + d_1 f_y + d_2 f_{yy} - E u_{xy} - G u_{xyy} - H u_{xxy} - K u_{xxyy}$ have been added. Here, α_h, α_k, c_1, c_2, d_1, d_2, E, G, H, K and F are given by (42).

3.2. $O(h^4 + k^4)$ compact exponential FD method: variable coefficients case

In this subsection, we will introduce an $O(h^4 + k^4)$ compact exponential FD approach for the model problem (31) with variable convection coefficients. Consider the FD scheme for Eq. (31) at a mesh point (x_i, y_j) as

$$\left[-\bar{\alpha}_h D_h^2 - \bar{\alpha}_k D_k^2 + c_{ij} D_h + d_{ij} D_k + \bar{E} D_h D_k + \bar{G} D_k^2 D_h + \bar{H} D_h^2 D_k + \bar{K} D_k^2 D_h^2\right] u_{ij} = \bar{F}_{ij} \tag{44}$$

where

$$\begin{aligned}
&\bar{\alpha}_h = \begin{cases} \frac{c_{ij}h}{2}\coth\left(\frac{c_{ij}h}{2a}\right), & c_{ij} \neq 0, \\ a, & c_{ij} = 0, \end{cases} \quad \bar{\alpha}_k = \begin{cases} \frac{d_{ij}k}{2}\coth\left(\frac{d_{ij}k}{2b}\right), & d_{ij} \neq 0 \\ b, & d_{ij} = 0 \end{cases} \\
&\bar{c}_1 = \begin{cases} \frac{a-\bar{\alpha}_h}{c_{ij}}, & c_{ij} \neq 0, \\ 0, & c_{ij} = 0, \end{cases} \quad \bar{c}_2 = \begin{cases} \frac{a(a-\bar{\alpha}_h)}{c_{ij}^2} + \frac{h^2}{6}, & c_{ij} \neq 0 \\ \frac{h^2}{12}, & c_{ij} = 0 \end{cases} \\
&\bar{d}_1 = \begin{cases} \frac{b-\bar{\alpha}_k}{d_{ij}}, & d_{ij} \neq 0, \\ 0, & d_{ij} = 0, \end{cases} \quad \bar{d}_2 = \begin{cases} \frac{b(b-\bar{\alpha}_k)}{d_{ij}^2} + \frac{k^2}{6}, & d_{ij} \neq 0 \\ \frac{k^2}{12}, & d_{ij} = 0 \end{cases} \\
&\bar{E} = \bar{c}_1 d_{ij} + \bar{d}_1 c_{ij}, \quad \bar{G} = \bar{d}_2 c_{ij} - \bar{c}_1 b, \quad \bar{H} = \bar{c}_2 d_{ij} - \bar{d}_1 a, \quad \bar{K} = -\bar{c}_2 b - \bar{d}_2 a \\
&\bar{F} = f + \bar{c}_1 f_x + \bar{c}_2 f_{xx} + \bar{d}_1 f_y + \bar{d}_2 f_{yy}
\end{aligned} \tag{45}$$

Using the Taylor series expansions and the original differential equation (31), we obtain the following modified partial differential equation corresponding to the scheme (44):

$$\begin{aligned}
&- a u_{xx} - b u_{yy} + c(x,y) u_x + d(x,y) u_y - (2\bar{c}_2 d_x + 2\bar{d}_2 c_y) u_{xy} - (2\bar{c}_2 c_x) u_{xx} - (2\bar{d}_2 d_y) u_{yy} \\
&\quad - (\bar{c}_1 c_x + \bar{c}_2 c_{xx} + \bar{d}_1 c_y + \bar{d}_2 c_{yy}) u_x - (\bar{c}_1 d_x + \bar{c}_2 d_{xx} + \bar{d}_1 d_y + \bar{d}_2 d_{yy}) u_y = f + O(h^4 + h^2 k^2 + k^4)
\end{aligned} \tag{46}$$

in which $\bar{c}_1$, $\bar{c}_2$, $\bar{d}_1$ and $\bar{d}_2$ are given by (45), and $\bar{c}_1 = -\frac{c_{ij}h^2}{12a} + O(h^4)$, $\bar{c}_2 = \frac{h^2}{12} + O(h^4)$, $\bar{d}_1 = -\frac{d_{ij}k^2}{12b} + O(k^4)$ and $\bar{d}_2 = \frac{k^2}{12} + O(k^4)$.

Eq. (46) shows that the local truncation error of Eq. (44) is only of $O(h^2 + k^2)$. To obtain an $O(h^4 + k^4)$ compact exponential FD scheme, adding the term $(2\bar{c}_2 d_x + 2\bar{d}_2 c_y) u_{xy} + (2\bar{c}_2 c_x) u_{xx} + (2\bar{d}_2 d_y) u_{yy} + (\bar{c}_1 c_x + \bar{c}_2 c_{xx} + \bar{d}_1 c_y + \bar{d}_2 c_{yy}) u_x + (\bar{c}_1 d_x + \bar{c}_2 d_{xx} + \bar{d}_1 d_y + \bar{d}_2 d_{yy}) u_y$ to the left-hand side of Eq. (31), we have

$$-A_f u_{xx} - B_f u_{yy} + C_f u_x + D_f u_y = F_p \tag{47}$$

where

$$\begin{aligned}
&A_f = a - 2\bar{c}_2 c_x, \quad C_f = c + \bar{c}_1 c_x + \bar{c}_2 c_{xx} + \bar{d}_1 c_y + \bar{d}_2 c_{yy} \\
&B_f = b - 2\bar{d}_2 d_y, \quad D_f = d + \bar{c}_1 d_x + \bar{c}_2 d_{xx} + \bar{d}_1 d_y + \bar{d}_2 d_{yy} \\
&F_p = f - (2\bar{c}_2 d_x + 2\bar{d}_2 c_y) u_{xy}
\end{aligned} \tag{48}$$

Eqs. (47) and (31) are the same in the form. Being similar to scheme (41), an $O(h^4+k^4)$ compact exponential FD scheme for the 2D convective diffusion model problem (31) with variable convection coefficients is given by

$$\left[-\tilde{A}D_h^2-\tilde{B}D_k^2+C_fD_h+D_fD_k+\tilde{E}D_hD_k+\tilde{G}D_k^2D_h+\tilde{H}D_h^2D_k+\tilde{K}D_k^2D_h^2\right]u_{ij}=\tilde{F}_{ij} \tag{49}$$

where

$$\begin{aligned}
&\tilde{A}=\begin{cases}\frac{C_fh}{2}\coth\left(\frac{C_fh}{2A_f}\right), & C_f\neq 0,\\ A_f, & C_f=0,\end{cases}\quad \tilde{B}=\begin{cases}\frac{D_fk}{2}\coth\left(\frac{D_fk}{2B_f}\right), & D_f\neq 0\\ B_f, & D_f=0\end{cases}\\
&\tilde{c}_1=\begin{cases}\frac{A_f-\tilde{A}}{C_f}, & C_f\neq 0,\\ 0, & C_f=0,\end{cases}\quad \tilde{c}_2=\begin{cases}\frac{A_f(A_f-\tilde{A})}{C_f^2}+\frac{h^2}{6}, & C_f\neq 0\\ \frac{h^2}{12}, & C_f=0\end{cases}\\
&\tilde{d}_1=\begin{cases}\frac{B_f-\tilde{B}}{D_f}, & D_f\neq 0,\\ 0, & D_f=0,\end{cases}\quad \tilde{d}_2=\begin{cases}\frac{B_f(B_f-\tilde{B})}{D_f^2}+\frac{k^2}{6}, & D_f\neq 0\\ \frac{k^2}{12}, & D_f=0\end{cases}\\
&\tilde{E}=\tilde{c}_1D_f+\tilde{d}_1C_f+2(\bar{c}_2d_{xij}+\bar{d}_2c_{yij})\\
&\tilde{G}=\tilde{d}_2C_f-\tilde{c}_1B_f,\quad \tilde{H}=\tilde{c}_2D_f-\tilde{d}_1A_f,\quad \tilde{K}=-\tilde{c}_2B_f-\tilde{d}_2A_f\\
&\tilde{F}=f+\tilde{c}_1f_x+\tilde{c}_2f_{xx}+\tilde{d}_1f_y+\tilde{d}_2f_{yy}
\end{aligned} \tag{50}$$

and $\bar{c}_2$, $\bar{d}_2$ and A_f, B_f, C_f, D_f are given by (45) and (48), respectively. In (50), A_f, B_f, C_f and D_f were written in short for A_{fij}, B_{fij}, C_{fij} and D_{fij}.

Scheme (49) with (50) is an $O(h^4+k^4)$ compact exponential FD scheme for the 2D model convective diffusion equation (31) on the nine-point 2D stencil. In Section 5, we shall present the numerical results that verify this inference.

The $O(h^4+k^4)$ compact exponential FD scheme (49) applied to Eq. (31) is equivalent to the standard central difference approximations applied to the following partial differential equation:

$$-\tilde{A}u_{xx}-\tilde{B}u_{yy}+C_fu_x+D_fu_y=\tilde{F}-\tilde{E}u_{xy}-\tilde{G}u_{xyy}-\tilde{H}u_{xxy}-\tilde{K}u_{xxyy} \tag{51}$$

Thus when the $O(h^4+k^4)$ FD scheme (49) is used, Eq. (31) is artificially perturbed to Eq. (51). Artificial diffusion coefficients $a\left[\frac{C_fh}{2a}\coth(\frac{C_fh}{2A_f})-1\right]$ and $b\left[\frac{D_fk}{2b}\coth(\frac{D_fk}{2B_f})-1\right]$, artificial convection coefficients $\bar{c}_1c_x+\bar{c}_2c_{xx}+\bar{d}_1c_y+\bar{d}_2c_{yy}$ and $\bar{c}_1d_x+\bar{c}_2d_{xx}+\bar{d}_1d_y+\bar{d}_2d_{yy}$, and an artificial source term $\tilde{c}_1f_x+\tilde{c}_2f_{xx}+\tilde{d}_1f_y+\tilde{d}_2f_{yy}-\tilde{E}u_{xy}-\tilde{G}u_{xyy}-\tilde{H}u_{xxy}-\tilde{K}u_{xxyy}$ have been added. Here, $\bar{c}_1$, $\bar{c}_2$, $\bar{d}_1$, $\bar{d}_2$, $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{d}_1$, $\tilde{d}_2$, $\tilde{A}$, $\tilde{B}$, C_f, D_f, $\tilde{E}$, $\tilde{G}$, $\tilde{H}$, $\tilde{K}$ and $\tilde{F}$ are as defined earlier.

Notice that Eqs. (43) and (51) are the same as Eq. (31) in form. Any efficient iterative method, such as ADI, splitting, SOR etc., used to solve Eq. (31) can also be easily applied to the fourth order Eqs. (43) and (51). Moreover, any existing code that solve the convection diffusion equation (31) with second order accuracy can be altered to provide fourth order accurate solutions just by adding some coefficients into the code. We also note that the resulting second order approximation for the left-hand side of Eq. (43) or (51) produce diagonally dominant matrix. Therefore, the line iterative or line iterative successive overrelaxtion (LSOR) approach with an alternating direction implicit (ADI) procedure enables us to obtain the solutions of the problems by application of the one-dimensional tridiagonal Thomas algorithm with a considerable saving in computing time.

4. Application to 2D incompressible Navier–Stokes equations

In this section, the proposed 4OC exponential FD formulation for the 2D convection–diffusion equation in Section 3 is used as the basis of a discretization method for the 2D incompressible Navier–Stokes equations using the stream function-vorticity ($\psi-\omega$) formulation, given by

$$-\psi_{xx} - \psi_{yy} = \omega \tag{52}$$

$$-\omega_{xx} - \omega_{yy} + ReU\omega_x + ReV\omega_y = 0 \tag{53}$$

$$U = \psi_y, \quad V = -\psi_x \tag{54}$$

where U and V are the velocities and Re is the nondimensional Reynolds number.

The stream function (52) is a Poisson equation, and the 4OC approximation can be given with $u = \psi, f = \omega$ and $c = d = 0$ in Eqs. (41) and (42). This 4OC FD scheme is

$$-\left[D_h^2 + D_k^2 + \left(\frac{h^2}{12} + \frac{k^2}{12}\right)D_k^2 D_h^2\right]\psi_{ij} = \left(1 + \frac{h^2}{12}D_h^2 + \frac{k^2}{12}D_k^2\right)\omega_{ij} \tag{55}$$

The velocity U at a grid point (x_i, y_j) can easily be calculated in the following way (also see [15] if $k = h$):

$$U_{ij} = \psi_{yij} = \left(D_k\psi - \frac{k^2}{6}\psi_{yyy}\right)_{ij} + O(k^4)$$

and using Eq. (52),

$$U_{ij} = D_k\psi_{ij} + \frac{k^2}{6}\left(\omega_y + \psi_{xxy}\right)_{ij} + O(k^4) = D_k\psi_{ij} + \frac{k^2}{6}\left(D_k\omega + D_h^2 D_k\psi\right)_{ij} + O(k^4 + h^2k^2) \tag{56}$$

Likewise for the y component of velocity V at a grid point (x_i, y_j)

$$V_{ij} = -D_h\psi_{ij} - \frac{h^2}{6}\left(D_h\omega + D_h D_k^2\psi\right)_{ij} + O(h^4 + h^2k^2) \tag{57}$$

The vorticity equation (53) is a special case of the 2D convection diffusion equation (31), and the 4OC approximation in this case may be obtained with $u = \omega$, $f = 0$, and $a = b = 1$, $c = ReU$, $d = ReV$ in Eqs. (45) and (48)–(50). According to Eqs. (48) and (50), the value of coefficients has to do with U, U_x, U_y, U_{xx}, U_{yy}, V, V_x, V_y, V_{xx} and V_{yy}, where $U_x = \psi_{xy}$, $U_y = \psi_{yy}$, $U_{xx} = \psi_{xxy}$, $V_x = -\psi_{xx}$, $V_y = -\psi_{xy}$ and $V_{yy} = -\psi_{xyy}$ can be approximated by the standard second-order central FD formulae. For a fully 4OC exponential FD scheme for (53), we need to approximate $U_{yy} = \psi_{yyy}$ and $V_{xx} = -\psi_{xxx}$ with $O(h^2 + k^2)$ accuracy on compact stencil, which is done as follows:

$$(U_{yy})_{ij} = (\psi_{yyy})_{ij} = -(\omega_y + \psi_{xxy})_{ij} = -\left(D_k\omega + D_h^2 D_k\psi\right)_{ij} + O(h^2 + k^2) \tag{58}$$

and

$$(V_{xx})_{ij} = -(\psi_{xxx})_{ij} = (\omega_x + \psi_{xyy})_{ij} = \left(D_h\omega + D_h D_k^2\psi\right)_{ij} + O(h^2 + k^2) \tag{59}$$

In Section 5, the newly proposed discretization method for the stream function-vorticity formulation of Navier–Stokes equations for incompressible viscous flow will be applied to obtain the numerical solutions of the lid driven cavity flow problem.

5. Numerical experiments

In this section, we perform numerical experiments to illustrate the accuracy, effectiveness and convergence of the HOC exponential FD schemes developed in this article. The numerical results of four linear and nonlinear problems, involving boundary layer problems, elliptic singular perturbation problems and the lid driven cavity flow problem, are given. For linear 1D problems, diagonally dominant tri-diagonal system of equations are directly solved and for nonlinear 1D problems, a line iterative or line iterative successive overrelaxtion (LSOR) procedure is associated with the solution of diagonally dominant tri-diagonal system of equations. For 2D problems, the line iterative or LSOR approach with an alternating direction implicit (ADI) procedure, which enables us to deal with only diagonally dominant tri-diagonal system of equations, is used to obtain the solutions of test problems. Comparisons are made between analytical solutions and numerical results for the currently proposed HOC exponential FD methods, as well as some previously published HOC methods. All computations are run on an SONY PCG-V505MCP computer using double precision arithmetic.

The iterative procedure is started with zero initial data and is terminated when $\mathrm{Max}|u_i^{m+1}-u_i^m|\leqslant 10^{-12}$ for one-dimensional problems and $\mathrm{Max}|u_{ij}^{m+1}-u_{ij}^m|\leqslant \delta$ for two-dimensional problems, where m is the iterative count.

The rate of convergence of each method is computed with the following definition:

$$\text{rate} = -\frac{\log(E^h/E^{h/2})}{\log(2)} \tag{60}$$

where E^h and $E^{h/2}$ are the absolute errors with the grids sizes h and $h/2$, respectively.

5.1. Problem 1

Consider the following differential equation in the presence of source term:

$$-\varepsilon u_{xx} + \frac{1}{1+x}u_x = f(x), \quad 0 < x < 1 \tag{61}$$

$$u(0) = 1 + 2^{-1/\varepsilon}, \quad u(1) = e + 2 \tag{62}$$

where $f(x)$ is given such that the exact solution is

$$u(x) = e^x + 2^{-1/\varepsilon}(1+x)^{1+1/\varepsilon} \tag{63}$$

This problem, which has a steep boundary layer at $x = 1$, is solved with the 4OC polynomial FD scheme [4,9,34], the perturbational 4OC exponential FD scheme proposed by Chen et al. [3], the 4OC exponential FD scheme proposed by Radhakrishna Pillai [27] and the present 4OC exponential FD scheme (20) for $\varepsilon = 10^{-3}$ and 10^{-5}.

Solution evolutions are shown in Figs. 1 and 2 with number of nodes *Nnode*. Notice that results computed by the present 4OC exponential FD method are in full agreement with exact results for all ε. For $\varepsilon = 10^{-3}$ and 10^{-5}, the present 4OC exponential FD method produces highly accurate monotone solutions on a 121-node discretization of the solution domain. For the modest value of $\varepsilon = 10^{-3}$, numerical solutions of the 4OC exponential FD schemes proposed by Radhakrishna Pillai [27] occur nonphysical oscillation and the results computed by the 4OC polynomial FD scheme [4,9,34] and Chen's perturbational 4OC exponential FD scheme [3] are not accurate on coarse mesh. Fig. 1 shows that 121 nodes are needed for the 4OC exponential FD scheme proposed by Radhakrishna Pillai [27] and the fourth-order compact polynomial FD scheme [4,9,34] to produce a acceptable solution. Note also that the results of Chen's perturbational h^4 exponential FD scheme [3] are not yet accurate for $Nnode = 221$. For the small value of $\varepsilon = 10^{-5}$, Radhakrishna Pillai's 4OC exponential FD method occur nonphysical oscillations for $Nnode = 221$ and at least 721 nodes are needed to produce a accurate nonoscillatory solution. The 4OC polynomial FD approximation, owing to overdiffusion, need a very fine mesh to accurately resolve solution gradients for convection dominated problems. It is seen from Fig. 2 that a twentyfold mesh refinement ($Nnode = 3601$) would be required for the 4OC polynomial FD method to produce comparable results, which only need 121-node for the present 4OC exponential FD scheme. Note that Chen's perturbational h^4 compact exponential FD scheme [3] are inaccurate for $Nnode = 12{,}001$ and at least 36,001 nodes are needed to produce a comparable solution.

5.2. Problem 2

Consider the following nonlinear differential equation, which has exact solution, in the absence of source term:

$$-\varepsilon u_{xx} + uu_x = 0, \quad -1 < x < 1 \tag{64}$$

for which the analytical solution is

$$u(x) = -\tanh\left(\frac{x}{2\varepsilon}\right) \tag{65}$$

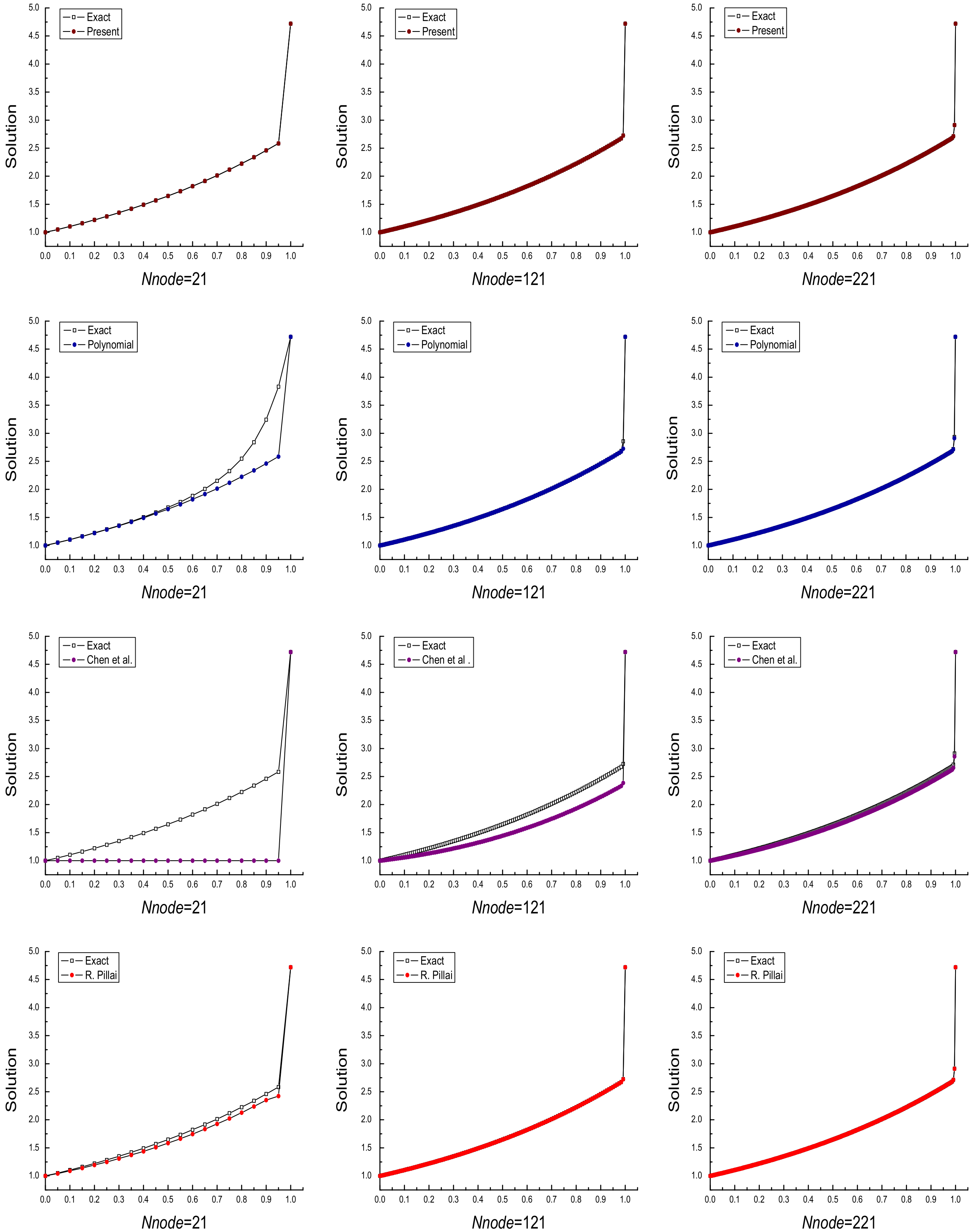


Fig. 1. 1D variable coefficient problem with nonzero source term at $\varepsilon = 10^{-3}$, solution dependence on *Nnode*. Computed solutions by the present 4OC exponential method, the 4OC polynomial method, the Radhakrishna Pillai's 4OC exponential method and the Chen's 4OC exponential method, *Problem 1*.

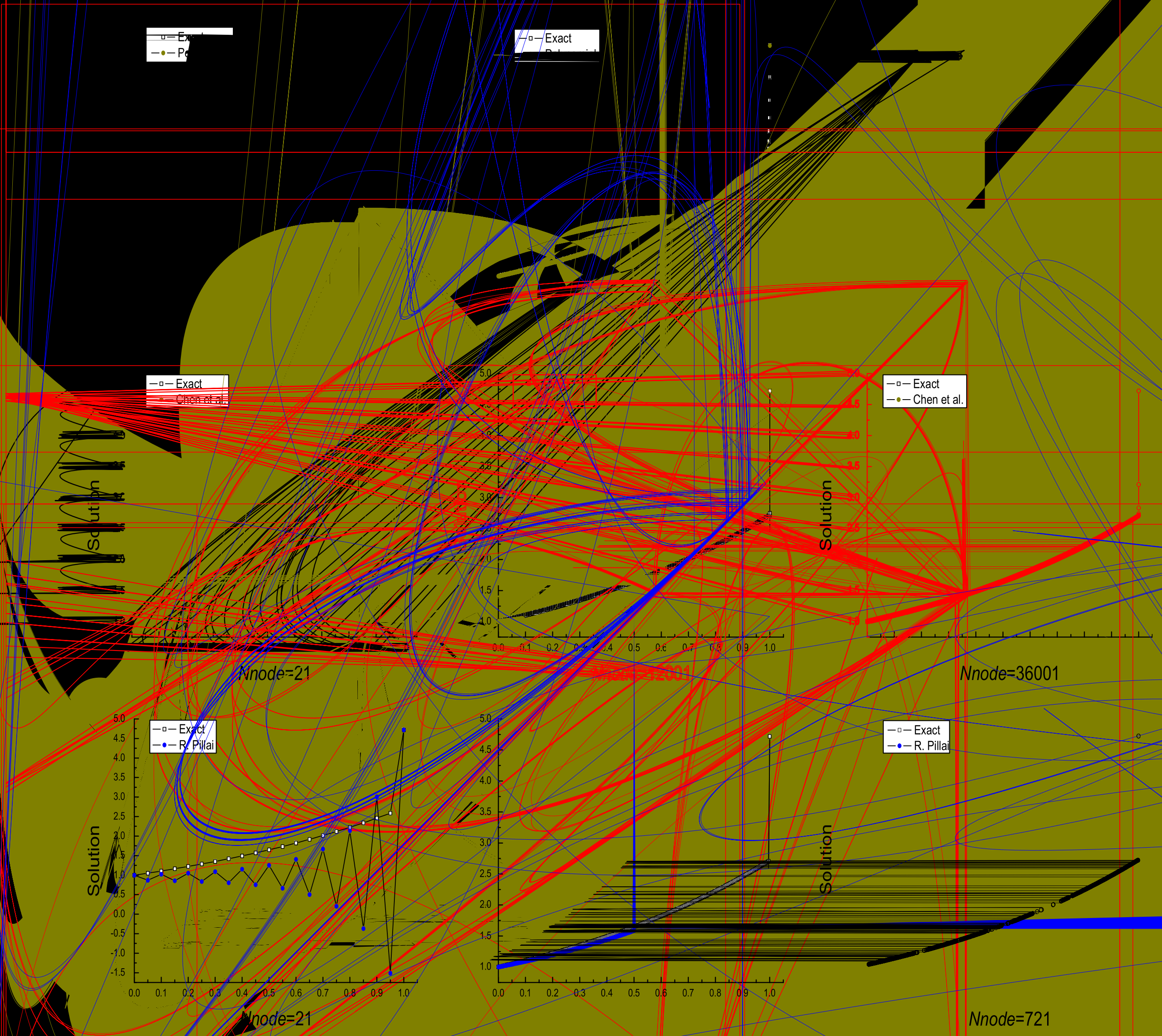


Fig. 2. 1D variable coefficient problem with nonzero source term at $\varepsilon = 10^{-5}$ ... dependence on *Nnode*. Computed solutions by the present 4OC exponential method, the 4OC polynomial method, ... 4OC exponential method and the Chen's 4OC exponential method, *Problem 1*.

The model problem (64) is the well-known steady Burgers equation originated from fluid flow research. In convection dominated cases with large values of the Reynolds number $Re = 1/\varepsilon$, the solution contains an abrupt change centred at the point $x = 0$, thus to model the nonlinear effects, such as the viscous boundary layer and shock wave, of fluid flow.

The calculations were carried out for $\varepsilon = 0.1$ using the 4OC polynomial FD scheme [34,4,9], the perturbational 4OC exponential FD scheme proposed by Chen et al. [3], the 4OC exponential FD scheme proposed by Radhakrishna Pillai [27] and the present 4OC exponential scheme (20). The approximation solutions at $x = -0.30$, absolute errors of these methods were given in Table 1. When compared to the exact solution $u(-0.30) = 0.90514825$ for $\varepsilon = 0.1$, the four $O(h^4)$ compact FD methods evidence superior performance. We also see that monotone and accurate results are obtained on a relatively coarse mesh. In addition, the convergence rates were computed by using (60). These values, which are also listed in Table 1, are approximately 4, confirming the $O(h^4)$ accuracy of these schemes.

In Fig. 3, solution evolutions for $Re(=1/\varepsilon) = 10^3$ are shown with number of nodes *Nnode*. Notice that the present 4OC exponential FD method produces highly accurate monotone solutions on a 121-node discretization of the solution domain. For the modest value of $\varepsilon = 10^{-3}$, numerical solution of the perturbational 4OC exponential FD scheme proposed by Chen et al. [3] is divergent on coarse mesh (*Nnode* = 21). Fig. 3 depicts that Chen's perturbational 4OC exponential FD scheme emerges inrealistic results for *Nnode* = 61 and Radhakrishna Pillai's 4OC exponential FD method occur nonphysical oscillations for *Nnode* = 221 and at least 421 nodes are needed to produce a nonoscillatory solution. The 4OC polynomial FD approximation, owing to overdiffusion, need a very fine mesh to accurately resolve solution gradients for convection dominated problems.

5.3. Problem 3

Consider the following differential equation in the presence of source term:

$$-\varepsilon(u_{xx} + u_{yy}) + \frac{1}{1+y}u_y = f(x,y), \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \tag{66}$$

with a Dirichlet boundary condition, where f is determined such that the analytic solution is

$$u(x,y) = e^{y-x} + 2^{-1/\varepsilon}(1+y)^{1+1/\varepsilon} \tag{67}$$

Table 1
Burger's equation with exact solution, $\varepsilon = 0.1$, *Problem 2*

Nnode	$u(x = -0.3)$	E	Rate
(a) Present 4OC exponential method			
21	0.90511805	0.30209×10^{-4}	–
41	0.90514674	0.15418×10^{-5}	4.292
81	0.90514816	0.90440×10^{-7}	4.092
(b) Chen's 4OC exponential method [3]			
21	0.90497145	0.17680×10^{-3}	–
41	0.90513703	0.11227×10^{-4}	3.977
81	0.90514755	0.70455×10^{-6}	3.994
(c) Radhakrishna Pillai's 4OC exponential method [27]			
21	0.90495837	0.18988×10^{-3}	–
41	0.90513600	0.12257×10^{-4}	3.953
81	0.90514748	0.77169×10^{-6}	3.989
(d) 4OC polynomial method [4,9,34]			
21	0.90474443	0.40382×10^{-3}	–
41	0.90512312	0.25129×10^{-4}	4.006
81	0.90514669	0.15678×10^{-5}	4.003
Exact solution	0.90514825		


Exact
R. Pillai
Exact
R. Pillai
Exact
R. Pillai

Table 2
Maximum absolute errors and the convergence rate, *Problem 3*

Nnode	4OC Polynomial method [4,9,34]	Chen 4OC method [3]	Pillai 4OC method [27]	Present 4OC method (49)
$\varepsilon = 1$				
21 × 21	0.11373(−07)	0.93680(−08)	0.11691(−07)	0.48972(−08)
41 × 41	0.71389(−09)	0.72035(−09)	0.73356(−09)	0.30815(−09)
81 × 81	0.45998(−10)	0.41478(−10)	0.47314(−10)	0.20296(−10)
Rate	3.956	4.118	3.951	3.924
$\varepsilon = 0.1$				
21 × 21	0.45349(−05)	0.25531(−05)	0.23105(−05)	0.17465(−05)
41 × 41	0.28267(−06)	0.16005(−06)	0.14413(−06)	0.10925(−06)
81 × 81	0.17657(−07)	0.10011(−07)	0.90105(−08)	0.68333(−08)
Rate	4.001	3.999	4.000	3.999
$\varepsilon = 0.01$				
21 × 21	0.33525(−01)	0.38620(−01)	0.24188(−02)	0.80604(−03)
41 × 41	0.28265(−02)	0.34746(−02)	0.17848(−03)	0.97768(−04)
81 × 81	0.16486(−03)	0.24211(−03)	0.11867(−04)	0.62914(−05)
Rate	4.100	3.843	3.911	3.958

Note: $0.11373(-07) = 0.11373 \times 10^{-7}$, etc.

This problem has a vertical boundary layer along $y = 1$. The calculations were carried out for $\delta = 10^{-14}$ using the current 4OC exponential FD scheme (49), the 4OC polynomial FD scheme developed by different authors in [5,9,17,34], the perturbational 4OC exponential FD scheme proposed by Chen et al. [3] and the 4OC exponential FD scheme proposed by Radhakrishna Pillai [27]. Maximum absolute errors are given in Table 2. Notice that the approximate solution from the present 4OC exponential FD method is more accurate than that from other 4OC exponential FD methods [3,27] and the 4OC polynomial FD method [5,17,9,34]. The estimated accuracy orders of the corresponding schemes with $h = 1/40$ and $h = 1/80$, which are computed by using (60), are also listed in Table 2. It can be seen that these HOC schemes attained their theoretical accuracy orders.

In Fig. 4, solution evolutions for $\varepsilon = 10^{-4}$ are shown with number of nodes *Nnode*. Notice that solution computed by the present 4OC exponential FD scheme (49) is in excellent agreement with exact results. The present 4OC exponential FD method produces highly accurate monotone solutions. Fig. 4 shows that the numerical solution of the 4OC exponential FD schemes proposed by Radhakrishna Pillai [27] occur nonphysical oscillation and the results computed by the 4OC polynomial FD scheme [5,9,17,34] and Chen's perturbational 4OC exponential FD scheme [3] are not accurate on coarse mesh.

5.4. Problem 4

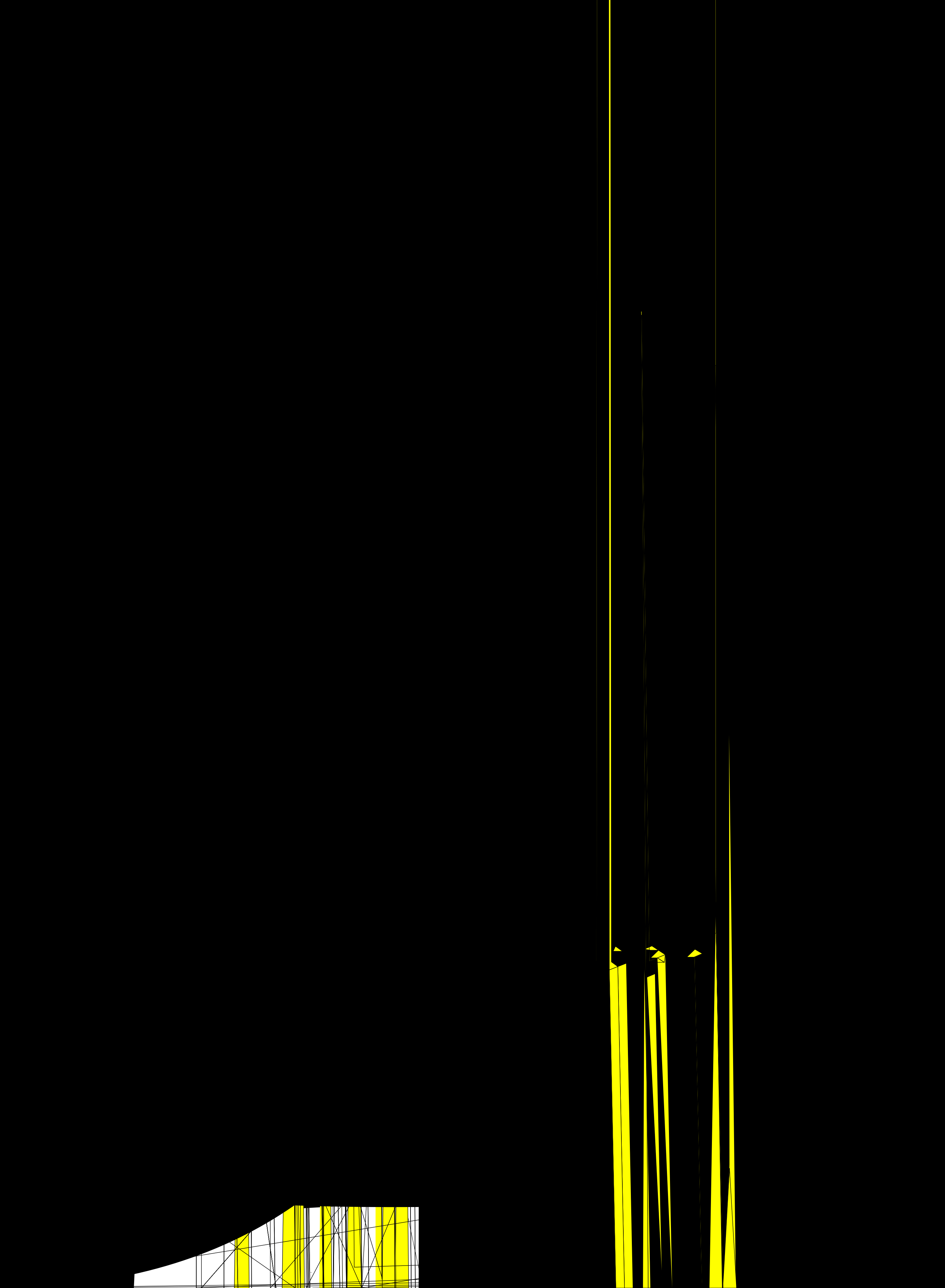
Consider the two-dimensional lid-driven cavity flow problem, which is extensively used as a benchmark for code validation of the incompressible Navier–Stokes equations. The cavity is defined in the square $[0,1] \times [0,1]$ and the governing equations are given by the stream function-vorticity formulation of the Navier–Stokes equations (52)–(54).

The flow is induced by the sliding motion of the top wall ($y = 1$) from left to right. The boundary conditions are those of no slip: on the top wall $U = 1$ and $V = 0$, on all other walls $U = 0$ and $V = 0$, as shown in Fig. 5. Further the stream function values on all four walls are zero ($\psi = 0$). In Fig. 5, the abbreviations BL, BR and TL refer to bottom left, bottom right and top left corners of the cavity, respectively.

The numerical boundary vorticity can be used by

$$\frac{h}{21}(6\omega_w + 4\omega_1 - \omega_2) + O(h^4) = \frac{1}{14h}(15\psi_w - 16\psi_1 + \psi_2) \pm V_w \tag{68}$$

given by Spotz [31], where V_w is the tangential wall velocity, 1 and 2 denote the first two neighbouring internal points on normal through the boundary w. For the driven cavity problem, $V_w = 0$ except on the moving lid, where $+V_w = U = 1$.



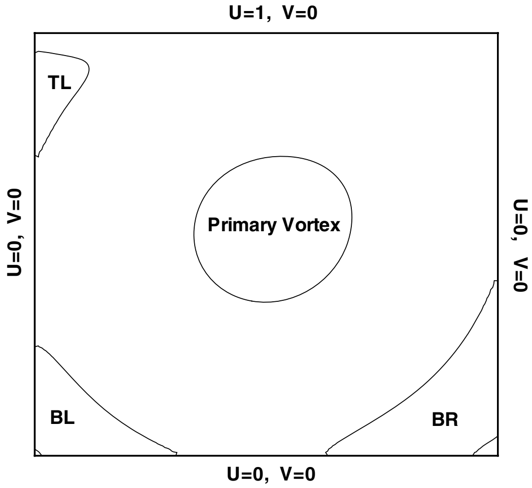


Fig. 5. Schematic of the vortex centers in the lid-driven cavity, *Problem 4*.

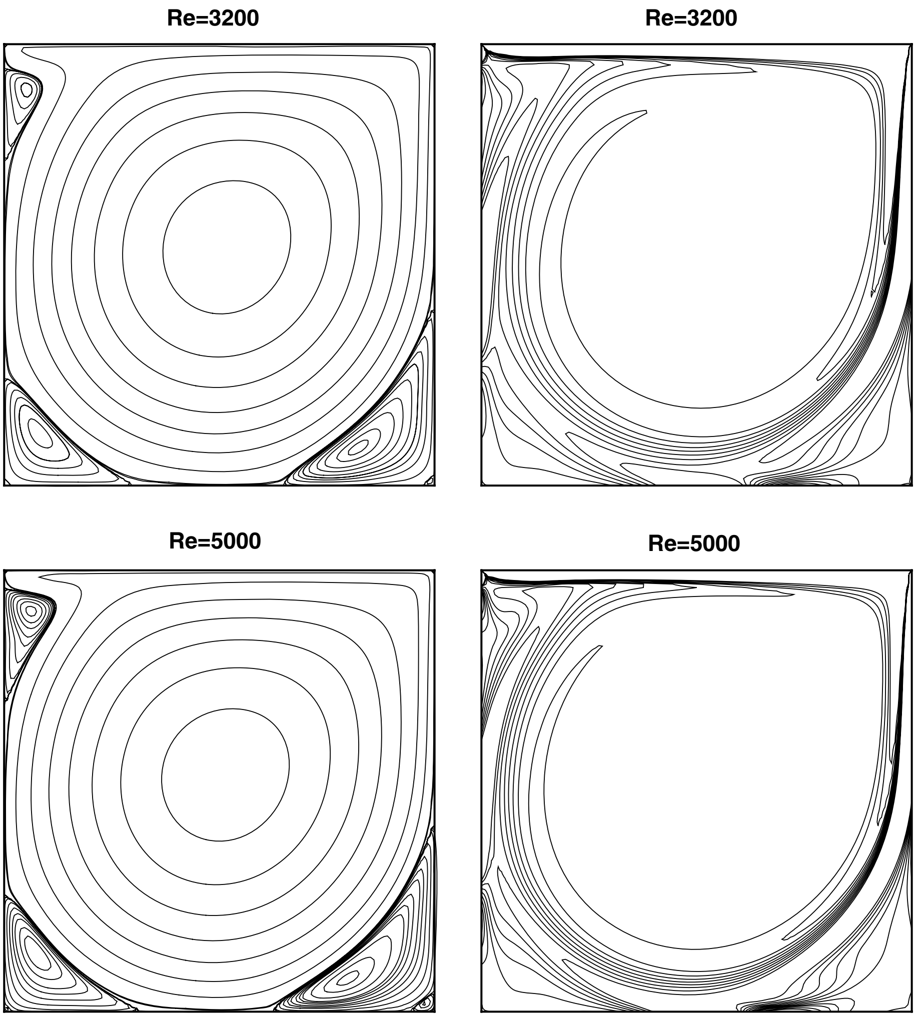


Fig. 6. Contours of streamfunction (left) and vorticity (right) with 129×129, *Problem 4*.

Table 3
Strength and location of the centers of primary vortex and secondary vortex BL and BR for the case with $Re = 1000$, 2000, 3200 and 5000

First author	Grid size	Primary vortex			Secondary vortex (BL1)		Secondary vortex (BR1)		Secondary vortex (TL)	
		ψ_{min}	ω	(x, y)	ψ_{max}	(x, y)	ψ_{max}	(x, y)	ψ_{max}	(x, y)
Re = 1000										
Present	65 × 65	−0.11791	−2.06082	(0.5313, 0.5625)	2.17e−4	(0.0781, 0.0781)	1.73e−3	(0.8594, 0.1094)	–	–
Present	129 × 129	−0.11882	−2.06719	(0.5313, 0.5625)	2.32e−4	(0.0859, 0.0781)	1.73e−3	(0.8594, 0.1094)	–	–
Ghia [14]	129 × 129	−0.11793	−2.04968	(0.5313, 0.5625)	2.31e−4	(0.0859, 0.0781)	1.75e−3	(0.8594, 0.1094)	–	–
Gupta [16]	81 × 81	−0.117	–	(0.5250, 0.5625)	2.02e−4	(0.0875, 0.0750)	1.70e−3	(0.8625, 0.1125)	–	–
Erturk [11]	601 × 601	−0.11894	−2.06776	(0.5300, 0.5650)	2.33e−4	(0.0833, 0.0783)	1.73e−3	(0.8633, 0.1117)	–	–
Re = 2000										
Present	65 × 65	−0.11957	−2.00607	(0.5156, 0.5625)	5.95e−4	(0.0938, 0.0938)	2.57e−3	(0.8281, 0.0938)	6.47e−4	(0.0313, 0.8750)
Present	129 × 129	−0.12063	−1.98823	(0.5234, 0.5469)	7.34e−4	(0.0859, 0.1016)	2.47e−3	(0.8359, 0.0938)	9.91e−5	(0.0313, 0.8750)
Gupta [16]	81 × 81	−0.118	–	(0.5250, 0.5500)	8.58e−4	(0.0875, 0.1000)	2.41e−3	(0.8375, 0.1000)	1.22e−4	(0.0375, 0.8875)
Re = 3200										
Present	129 × 129	−0.12089	−1.95085	(0.5156, 0.5391)	1.06e−3	(0.0781, 0.1172)	2.83e−3	(0.8203, 0.0859)	6.76e−4	(0.0547, 0.8984)
Ghia [14]	129 × 129	−0.12038	−1.98860	(0.5165, 0.5469)	9.78e−4	(0.0859, 0.1094)	3.14e−3	(0.8125, 0.0859)	7.28e−4	(0.0547, 0.8984)
Li [22]	129 × 129	−0.12053	−1.94286	–	–	–	–	–	–	–
Gupta [16]	161 × 161	−0.122	–	(0.5188, 0.5488)	1.03e−3	(0.0813, 0.1188)	2.86e−3	(0.8125, 0.0875)	7.33e−4	(0.0563, 0.9000)
Re = 5000										
Present	129 × 129	−0.12070	−1.92215	(0.5156, 0.5391)	1.26e−3	(0.0703, 0.1328)	3.10e−3	(0.8047, 0.0781)	1.37e−3	(0.0625, 0.9063)
Ghia [14]	257 × 257	−0.11897	−1.86015	(0.5117, 0.5352)	1.36e−3	(0.0703, 0.1367)	3.08e−3	(0.8086, 0.0742)	1.46e−3	(0.0625, 0.9102)
Li [22]	129 × 129	−0.12036	−1.92430	–	–	–	–	–	–	–
Gupta [16]	161 × 161	−0.122	–	(0.5125, 0.5375)	1.32e−3	(0.0750, 0.1313)	2.96e−3	(0.8000, 0.0750)	1.54e−3	(0.0688, 0.9125)
Erturk [11]	601 × 601	−0.12222	−1.94055	(0.5150, 0.5350)	1.38e−4	(0.0733, 0.1367)	3.07e−3	(0.8050, 0.0733)	1.45e−3	(0.0633, 0.9100)

the bottom corners of the cavity as well as TL at the top left of the square cavity. In Fig. 6, the evolution of tertiary vortices in the bottom corners of the cavity are also observed. The tertiary vortices become visible for $Re = 3200$ and gain a significant size for $Re = 5000$.

The location coordinates of the centre of primary and secondary vortices and the corresponding stream function and/or vorticity values at the centre of these vortices are summarized in Table 3 for $Re = 1000$, 2000, 3200 and 5000. The available comparison data from the literature are also given in these tables. In each case our results exhibit an excellent match with the best and most accurate solutions available in the literature.

6. Concluding remarks

In this article, we have developed HOC exponential, referred to as EHOC, FD methods for the solution of one- and two-dimensional convection–diffusion equations with constant and variable convection coefficients. A distinguishing desirable property of the developed method is solution matrix bandwidth, which always remains equal to that of the second-order discretizations. For the 1D convection–diffusion model problems, the difference equations are proposed on the three-point 1D stencil, which result in diagonally dominant tri-diagonal systems. For the 2D case, the EHOC schemes on the nine-point 2D stencil are derived. The simple line iterative or line iterative SOR technique with an alternating direction implicit (ADI) procedure enables one to deal with only diagonally dominant tri-diagonal systems which can be solved by application of the one-dimensional tridiagonal Thomas algorithm with a considerable saving in computing time. This permits combining the computational efficiency of the lower order methods with superior accuracy inherent in high order approximations. As the basis of a discretization method for the incompressible, 2D, steady-state flow problems, the 4OC exponential FD formulation proposed for the 2D convection–diffusion equation has been extended to the stream function-vorticity formulation of the Navier–Stokes equations. The present method is easily extendible to three-dimensional convection–diffusion type problems.

Numerical experiments are performed to demonstrate their high accuracy and efficiency and to show their superiority over the HOC polynomial FD scheme [4,5,9,17,34], the perturbational 4OC exponential FD scheme proposed by Chen et al. [3] and the 4OC exponential FD scheme proposed by Radhakrishna Pillai [27], in terms of resolution of solution gradients. The robustness of the present EHOC methods is illustrated by their applicability to the wide range problems including linear and nonlinear problems, mostly with small second derivative terms, in particular, fluid flow problems with boundary or internal layers. The computational results show that, besides including the excellent performances of the HOC polynomial method in computational accuracy, efficiency and stability, the present EHOC method has the advantage of better scale resolution with smaller number of gird nodes.

The theoretical analysis and experiments studies of the convergence and performance of iterative methods with HOC schemes should be beneficial. The convergence and performance of iterative methods with HOC polynomial schemes have been studied in [18,38,39]. The convergence and performance of iterative methods with EHOC schemes have not yet to be investigated. This is the subject of the future.

Finally, we mention that the usage of nonuniform mesh becomes an essential aspect for dealing with convection-dominated problems [12,13]. However, the present EHOC schemes are all based on the uniform mesh-size discretization. We are currently working to develop an EHOC FD method that is suitable for both uniform and nonuniform grids.

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Appendix 1

This appendix present some useful definitions and theorems [25,35].

Definition 1. A real $n \times n$ matrix $A = (a_{ij})$ is said to be irreducibly diagonally dominant if $|a_{ii}| \geqslant \sum_{j=1, j\neq i}^{n} |a_{ij}|$ for all $0 \leqslant i \leqslant n$, and the strict inequality holds for at least one i.

Definition 2. A real $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \leqslant 0$ for $i \neq j$ and $a_{ii} > 0$ for $1 \leqslant i \leqslant n$ is an M-matrix if A is nonsingular and its inverse has no negative entries (or $A^{-1} \geqslant 0$).

Theorem 1. *If $A = (a_{ij})$ is a real, strictly or irreducible diagonally dominant $n \times n$ matrix with $a_{ij} \leqslant 0$ for $i \neq j$ and $a_{ii} > 0$ for $1 \leqslant i \leqslant n$, then A is invertible and $A^{-1} > 0$, which means that A is an M-matrix.*

Definition 3. A real $n \times n$ matrix $A = (a_{ij})$ is defined to be monotone if $A\phi \geqslant 0$ holds for any vector ϕ, it implies $\phi \geqslant 0$.

Theorem 2. *If the off-diagonal entries of A are nonpositive, we are led to a monotone A if and only if A is an M-matrix.*

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